

THE RATIONAL SECTIONAL CATEGORY OF CERTAIN UNIVERSAL FIBRATIONS

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ABSTRACT. We prove that the sectional category of the universal fibration with fibre X , for X any space that satisfies a well-known conjecture of Halperin, equals one after rationalization.

1. MAIN RESULT

We begin with a concise résumé of the ingredients, then a statement, of our main result. Notations used and assertions made here are described in greater detail below. *Sectional category* (\mathbf{secat}) is a numerical invariant of a fibration that extends the notion of *LS category* (\mathbf{cat}) of a space to fibrations. We normalize these invariants: $\mathbf{secat}(p) = 0$ when fibration $p: E \rightarrow B$ has a section. Sectional category plays a role in several interesting applications (e.g. see [1, §9.3] and [4]).

Fibrations with fibre a fixed space X are classified by a universal fibration with fibre X [17, 2, 14]. For fibrations of simply connected spaces, this universal fibration may be identified, up to homotopy, as the map on Dold-Lashof classifying spaces $u_X: \mathbf{Baut}_1^*(X) \rightarrow \mathbf{Baut}_1(X)$ that is induced by an inclusion $\mathbf{aut}_1^*(X) \hookrightarrow \mathbf{aut}_1(X)$ of connected monoids of self-equivalences [3, 9].

A fibration $p: E \rightarrow B$ with B simply connected admits a rationalization $p_{\mathbb{Q}}: E_{\mathbb{Q}} \rightarrow B_{\mathbb{Q}}$ that, homotopically, represents a simplification of p . Then the *rational sectional category* (\mathbf{secat}_0) of fibration p is defined by setting $\mathbf{secat}_0(p) := \mathbf{secat}(p_{\mathbb{Q}})$.

Now we have general inequalities $\mathbf{secat}_0(p) \leq \mathbf{secat}_0(u_X) \leq \mathbf{cat}_0(\mathbf{Baut}_1(X))$, with $\mathbf{cat}_0(\mathbf{Baut}_1(X))$ the *rational LS category* of the classifying space, which is often infinite (see [6, 7]). So it is natural to ask whether $\mathbf{secat}_0(u_X)$ may be finite.

Our main result is the following:

Theorem 1.1. *Let X be any F_0 -space that satisfies Conjecture 1 below. Then $\mathbf{secat}_0(u_X) = 1$, where $u_X: \mathbf{Baut}_1^*(X) \rightarrow \mathbf{Baut}_1(X)$ denotes the universal fibration with fibre X .*

An F_0 -space is any (non-trivial) space with $H^*(X; \mathbb{Q})$ and $\pi_*(X) \otimes \mathbb{Q}$ both finite-dimensional and $H^{\text{odd}}(X; \mathbb{Q}) = 0$. Examples of F_0 -spaces for which Conjecture 1 is satisfied include even-dimensional spheres, complex projective spaces, homogeneous spaces G/H with $\text{rank } G = \text{rank } H$, and finite products of any of these spaces.

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2. INTRODUCTION

We continue with a fuller description of the ingredients just indicated. We assume all spaces are simply connected CW complexes of finite type. Let $p: E \rightarrow B$ be a fibration. For $n \geq 1$, set $\text{secat}(p) = n - 1$ if n is the minimal number of open sets U_i in a cover of B such that p admits a section over each U_i . Set $\text{cat}(B) = n - 1$ if n is the minimal number of open sets U_i in a cover of B such that each inclusion $U_i \rightarrow B$ is nullhomotopic. The inequalities $\text{secat}(p) \leq \text{cat}(B)$ and $\text{secat}(f^*(p)) \leq \text{secat}(p)$, for $f^*(p)$ the pull-back of p by a map $f: B' \rightarrow B$, are both proved directly (see, e.g. Proposition 9.14 and Exercise 9.3 of [1], which reference also contains many facts concerning secat and cat). Generally speaking, both cat and secat are delicate invariants, and difficult to compute. It is quite surprising, therefore, that we are able to obtain a global result such as Theorem 1.1. This is especially so considering that the universal fibration is a rich construction that involves large and complex spaces whose general structure is not well understood.

Because $u_X: \text{Baut}_1^*(X) \rightarrow \text{Baut}_1(X)$ is *universal*, $\text{secat}(u_X)$, when finite, is an upper bound for the sectional category of every fibration with fibre X . From properties of rationalization, we have $\text{secat}_0(p) \leq \text{secat}(p)$. This inequality follows using an alternate characterization of secat that we indicate at the start of the next section—involving the so-called fibrewise join construction. We also have $\text{secat}_0(p) := \text{secat}(p_{\mathbb{Q}}) \leq \text{cat}(B_{\mathbb{Q}})$, with the inequality following as before, applied to the rationalized fibration $p_{\mathbb{Q}}$. For simply connected B , we have $\text{cat}(B_{\mathbb{Q}}) = \text{cat}_0(B)$, where cat_0 denotes the rational LS category. By the naturality of rationalization with respect to pull-backs, we obtain $\text{secat}_0(p) \leq \text{secat}_0(u_X)$ for any fibration $p: E \rightarrow B$ with fibre X .

Stanley has given a complete calculation of the rational sectional category of spherical fibrations [16]. His results for the even-dimensional sphere imply (in our normalized notation) that $\text{secat}_0(u_{S^{2n}}) = 1$. Here, we extend Stanley's result from S^{2n} to any F_0 -space that satisfies Halperin's Conjecture in rational homotopy theory. We discuss this class of spaces now.

As above, an F_0 -space X is an *elliptic space*— $H^*(X; \mathbb{Q})$ and $\pi_*(X) \otimes \mathbb{Q}$ are both finite-dimensional—with evenly-graded rational cohomology: $H^{\text{odd}}(X; \mathbb{Q}) = 0$. Halperin has conjectured the following generalization of classical results on the rational cohomology of homogeneous spaces.

Conjecture 1 (Halperin). *Let X be an F_0 -space and $p: E \rightarrow B$ any fibration of simply connected spaces with fibre X . Then the rational Serre spectral sequence for p collapses at the E_2 -term.*

The conjecture is equivalent to the assertion that $\text{Der}_{>0}(H^*(X; \mathbb{Q})) = 0$ for X an F_0 -space where $\text{Der}_{>0}(A)$ is the graded Lie algebra of degree-lowering derivations of the algebra A [18, 15]. The conjecture follows easily from this version for $X = S^{2n}, \mathbb{C}P^n$ and, more generally, for any space with rational cohomology a truncated polynomial algebra. Meier proved the conjecture for flag-manifolds G/T with T a maximal torus, and other homogeneous spaces [15, Th.B]. Shiga and Tezuka extended Meier's result to the general case of homogeneous spaces G/H of equal rank pairs [17]. Halperin's Conjecture has also been confirmed for the cases in which $H^*(X; \mathbb{Q})$ has 3 or fewer generators [12]. Markl has shown that the class of spaces for which Conjecture 1 holds is closed under fibrations, not just products

[13]. Our main result, Theorem 1.1, applies in all these cases. We refer the reader to [5, p.516] for a discussion and other references.

Meier made the following connection between Halperin's Conjecture and the rational homotopy of the universal fibration [15, Th.A]

Theorem 2.1 (Meier). *Let X be an F_0 -space. Then X satisfies Halperin's Conjecture if and only if $Baut_1(X)$ is rationally equivalent to a product of even-dimensional Eilenberg-Mac Lane Spaces.* \square

Meier's Theorem implies that $\text{cat}_0(Baut_1(X)) = \infty$ for X an F_0 -space that satisfies Halperin's Conjecture. In fact, this is the case for any elliptic space X [6]. We will use Theorem 2.1 to deduce Theorem 1.1 in Section 3 as a consequence of Proposition 3.2 a technical result concerning sections of rational fibrations.

3. RATIONAL SECTIONAL CATEGORY OF A FIBREWISE JOIN

We recall a special case of a characterization of secat in terms of the fibrewise join construction. Given $p: E \rightarrow B$ with fibre X , the fibrewise join $p * p$ is a fibre sequence: $X * X \longrightarrow E * E \xrightarrow{p * p} B$. As is well-known, we may identify the (ordinary) join as $X * X = \Sigma(X \wedge X)$. The following is a special case of a result of Svarc (see [11, Prop.8.1]):

Proposition 3.1. *Let $p: E \rightarrow B$ be a fibration. Then $\text{secat}(p) \leq 1$ if and only if $p * p$ has a section.* \square

Our main result is that the fibrewise join of the universal fibration

$$X * X \longrightarrow Baut_1^*(X) * Baut_1^*(X) \xrightarrow{u_X * u_X} Baut_1(X)$$

has a section after rationalization when X is an F_0 -space that satisfies Halperin's Conjecture. We make use of the correspondence between fibre sequences of rational spaces and relative Sullivan models. Although we will recall some basic facts about minimal models, our proofs assume a working familiarity with them. Our reference for rational homotopy is [5].

Let $X \rightarrow E \xrightarrow{p} B$ be a fibre sequence. The *relative Sullivan model* for p is a sequence

$$\wedge W, d_B \xrightarrow{J} \wedge W \otimes \wedge V, D \longrightarrow \wedge V, d_X$$

of DG algebras, with $\wedge W, d_B$ and $\wedge V, d_X$ the Sullivan minimal models for B and X , respectively [5, Prop.15.5]. The differential D satisfies

$$D(w) = d_B(w) \text{ for } w \in W \text{ and } D(v) - d_X(v) \in \wedge^+ W \otimes \wedge V \text{ for } v \in V.$$

The inclusion J is a model for p . Applying spatial realization, we obtain that $p_{\mathbb{Q}}: E_{\mathbb{Q}} \rightarrow B_{\mathbb{Q}}$ admits a section if and only if J has a left-inverse S . That is, $p_{\mathbb{Q}}$ admits a section if and only if there is a DG algebra map $S: \wedge W \otimes \wedge V, D \rightarrow \wedge W, d_B$ with $S \circ J = \text{id}_{\wedge W}$. We prove:

Proposition 3.2. *Let $X \rightarrow E \xrightarrow{p} B$ be a fibre sequence. Suppose*

- (1) *B has the rational homotopy type of a product of even-dimensional Eilenberg-Mac Lane spaces, and*
- (2) *X has the rational homotopy type of a wedge of at least two odd-dimensional spheres.*

Then the rationalization $p_{\mathbb{Q}}: E_{\mathbb{Q}} \rightarrow B_{\mathbb{Q}}$ of p admits a section.

Proof. In the following, we make use of the identification $V^n \cong \text{Hom}(\pi_n(X), \mathbb{Q})$, where $(\wedge V, d)$ is the minimal model of X , and also the identification between Samelson products in the rational homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ and the quadratic part of the differential in $(\wedge V, d)$, for any space X . See [5, Th.15.11, Th.21.6] for details. Hypothesis (1) implies that the minimal model $\wedge W, d_B$ for B has trivial differential, $d_B = 0$, with $W^{\text{odd}} = 0$. We deduce two consequences of (2) for the minimal model of the fibre, $\wedge V, d_X$. First we see that $V^{\text{even}} = 0$ (a wedge of odd-dimensional spheres has no non-zero rational homotopy in even degrees). Write $V = \langle v_1, v_2, \dots \rangle$ with $|v_i| \leq |v_j|$ whenever $i < j$ and each $|v_i|$ odd. Then, for degree reasons, we have $d_X(v_1) = d_X(v_2) = 0$. The rational homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ is free as graded Lie algebra. Translated to Sullivan models, we deduce, in particular, that there is some v_k for $k \geq 3$, such that $d_F(v_k) = v_1 v_2$.

The relative Sullivan model for $X \rightarrow E \rightarrow B$ is of the form:

$$\wedge W, 0 \rightarrow \wedge W \otimes \wedge V, D \rightarrow \wedge V, d_X.$$

Given a subspace $V' \subseteq V$, let $I(V') = \wedge W \otimes \wedge^+ V'$ denote the ideal generated by V' in $\wedge W \otimes \wedge V$. Our goal is to prove that $I(V)$ is a D -stable ideal of $\wedge W \otimes \wedge V$. We may then define a DG algebra map $S: \wedge W \otimes \wedge V, D \rightarrow \wedge W, 0$ by $S(w) = w$ and $S(v) = 0$ to obtain the desired section.

We use induction on i to show that, for any v_i , we have $D(v_i) \in I(V)$. First, we show that $D(v_1) = 0$. For suppose that $D(v_1) = P$, for some polynomial $P \in \wedge W$. Since $d_X(v_1) = d_X(v_2) = 0$, we see that $D(v_2) = P_2$ for some $P_2 \in \wedge W$. Then, with v_k chosen as above, we have $D(v_k) = v_1 v_2 + P_k$, for some $P_k \in \wedge W$. Notice that there cannot be a term from $\wedge^+ W \otimes \wedge^+ V$ in $D(v_k)$, since V is oddly graded and $D(v_k)$ is of even degree, and also because $v_1 v_2$ is of minimal degree in $\wedge^2 V$. Furthermore, since $v_1 v_2$ is the only term from $\wedge^2 V$ appearing in $d(v_k)$, $v_1 v_2$ is the only such term appearing in $D(v_k)$. Since $D^2(v_k) = 0$, we have

$$0 = D^2(v_k) = D(v_1 v_2 + P_k) = P v_2 - v_1 P_2,$$

and it follows that $P = 0$ (also that $P_2 = 0$).

Now suppose that we have $D(v_i) \in I(V)$ for all $i < t$ and for some $t \geq 2$. Then $I(v_1, \dots, v_{t-1})$ is a D -stable ideal of $\wedge W \otimes \wedge V$. We may take the quotient by this ideal yielding the graded algebra

$$\wedge W \otimes \wedge V_{i \geq t} = (\wedge W \otimes \wedge V) / I(v_1, \dots, v_{t-1}).$$

Since $I(v_1, \dots, v_{t-1})$ is D -stable, D induces a differential \overline{D} on the quotient. Let $\pi: \wedge W \otimes \wedge V, D \rightarrow \wedge W \otimes \wedge V_{i \geq t}, \overline{D}$ denote the projection which is a map of DG algebras.

Next observe that $\wedge(v_1, \dots, v_{t-1})$ is a d_X -stable sub-algebra of $\wedge V, d_X$. Write $\overline{d_X}$ for the induced differential and $\pi': \wedge V, d_X \rightarrow \wedge V_{i \geq t}, \overline{d_X}$ for the projection. We claim that $\wedge V_{i \geq t}, \overline{d_X}$ is the minimal model for a wedge of odd-dimensional spheres. For observe that π' is surjective on generators. Applying the Mapping Theorem [5, Th.29.5], we deduce that

$$\text{cat}(\wedge V_{i \geq t}, \overline{d_X}) \leq \text{cat}(\wedge V, d_X) = \text{cat}_0(X) = 1.$$

Our claim follows (recall that V , hence $V_{i \geq t}$, is concentrated in odd degrees).

Now consider the commutative diagram of relative Sullivan models:

$$\begin{array}{ccccc}
 \wedge W, 0 & \longrightarrow & \wedge W \otimes \wedge V, D & \longrightarrow & \wedge V, d_X \\
 \parallel & & \downarrow \pi & & \downarrow \pi' \\
 \wedge W, 0 & \longrightarrow & \wedge W \otimes \wedge V_{i \geq t}, \overline{D} & \longrightarrow & \wedge V_{i \geq t}, \overline{d}_X.
 \end{array}$$

The bottom sequence of this diagram corresponds to a fibre sequence with the original base, say $\overline{X}_{\mathbb{Q}} \rightarrow \overline{E}_{\mathbb{Q}} \rightarrow B_{\mathbb{Q}}$. We have argued above that the fibre $\overline{X}_{\mathbb{Q}}$, with minimal model $\wedge V_{i \geq t}, \overline{d}_X$, is a wedge of odd-dimensional spheres. We can now apply the first part of the argument above, to deduce that $\overline{D}v_t = 0$. But this implies that $D(v_t) \in I(v_1, \dots, v_{t-1}) \subseteq I(V)$. By induction, we conclude that $I(V)$ is D -stable. \square

Examples 3.3. We allustrate that Proposition 3.2 is a sharp result, at least if we consider rational fibrations with fibre a wedge of spheres and base a product of Eilenberg-Mac Lane spaces.

- (a) We must have only even-dimensional Eilenberg-Mac Lane spaces in the base. For consider $\Omega S^3 * \Omega S^3 \rightarrow S^3 \vee S^3 \xrightarrow{p} S^3 \times S^3$ obtained by converting the inclusion to a fibration. Here, the fibre $\Omega S^3 * \Omega S^3$ is rationally a wedge of odd-dimensional spheres. We can see that $p_{\mathbb{Q}}$ does not admit a section, however, since $p^*: H^*(S^3 \times S^3; \mathbb{Q}) \rightarrow H^*(S^3 \vee S^3; \mathbb{Q})$ is not injective.
- (b) We must have a wedge of at least two odd-dimensional spheres in the fibre. For example, the path-loop fibration $K(\mathbb{Q}, 2k-1) \rightarrow PK(\mathbb{Q}, 2k) \rightarrow K(\mathbb{Q}, 2k)$ does not have a section. In fact, since the total space here is contractible we have $\text{secat}_0(p) = \text{cat}(K(\mathbb{Q}, 2k)) = +\infty$. Recall that we have $S^{2k-1} \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2k-1)$.
- (c) We cannot allow even-dimensional spheres in the fibre. For suppose E' is a wedge of spheres and set $E = S^{2k} \vee E'$. By pinching off E' , then including the sphere as the bottom cell of an Eilenberg-Mac Lane space, we obtain a fibre sequence $X \rightarrow E \xrightarrow{p} K(\mathbb{Q}, 2k)$ with fibre X a wedge of spheres (of both odd and even dimensions). This fibration cannot admit a section, as p^* is not injective in cohomology.

We next observe that the universal fibration u_X for X a nontrivial elliptic space does not admit a section. To prove this, we use the *Gottlieb group* $G_*(X) \subseteq \pi_*(X)$ [8]. Recall that $G_*(X) = \text{Image}\{\omega_{\sharp}: \pi_*(\text{aut}_1(X)) \rightarrow \pi_*(X)\}$ where $\omega: \text{aut}_1(X) \rightarrow X$ is the evaluation map.

Proposition 3.4. *Let X be a nontrivial elliptic space, such as any F_0 -space. Then $\text{secat}_0(u_X) \geq 1$.*

Proof. First note that we have $G_*(X_{\mathbb{Q}}) \neq 0$ for any elliptic space X . This is because, in the minimal model $(\wedge V, d)$ for X , there is an (odd) integer $n > 0$ such that $V^n \neq 0$ and $V^m = 0$ for $m > n$. Then it follows from the identification of the Gottlieb group in minimal model terms, discussed in [5, §29d], that we have $G_n(X_{\mathbb{Q}}) \cong V^n \neq 0$. Next, by [8, §4], $G_*(X)$ corresponds to the image of $\partial: \pi_{*+1}(B\text{aut}_1(X)) \rightarrow \pi_*(X)$, the linking homomorphism in the long exact sequence of the universal fibration. Thus $G_*(X_{\mathbb{Q}}) \neq 0$ implies that $(u_X)_{\mathbb{Q}}$ does not induce a surjection on rational homotopy groups and so cannot admit a section. \square

We apply the preceding to prove our main result:

Proof of Theorem 1.1. Let X be a nontrivial F_0 -space. By Proposition 3.4, we have $\text{secat}_0(u_X) \geq 1$. We prove $\text{secat}_0(u_X) \leq 1$.

The fibrewise join of the universal fibration, $u_X * u_X$, has base $B\text{aut}_1(X)$ and fibre $X * X = \Sigma(X \wedge X)$. By Theorem 2.1, if X satisfies Halperin's Conjecture then $B\text{aut}_1(X)$ is rationally a product of even-dimensional Eilenberg-Mac Lane spaces. Regarding the fibre, note that, since $H^*(X; \mathbb{Q})$ is evenly graded, $H^*(\Sigma(X \wedge X); \mathbb{Q})$ is oddly graded. By [10, Th.1.5], $X * X$ has the rational homotopy type of a wedge of odd-dimensional spheres. If $\tilde{H}^*(X; \mathbb{Q})$ has dimension at least 2, then $X * X$ is rationally a wedge of at least two odd-dimensional spheres. Applying Proposition 3.2, we conclude the rationalization of $u_X * u_X$ has a section and we conclude $\text{secat}_0(u_X) \leq 1$ by Proposition 3.1.

When $\tilde{H}^*(X; \mathbb{Q})$ has dimension 1, then $X \simeq_{\mathbb{Q}} S^{2n}$ and we can invoke Stanley's result [16, Lem.3.1]. Alternately, we may observe that the fibrewise join $u_{S^{2n}} * u_{S^{2n}}$ has fibre $S^{2n} * S^{2n} \simeq_{\mathbb{Q}} K(\mathbb{Q}, 4n - 1)$ and base $B\text{aut}_1(S^{2n}) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 4n)$. It follows easily from degree considerations that $(u_{S^{2n}} * u_{S^{2n}})_{\mathbb{Q}}$ is fibre-homotopically trivial and so, in particular, has a section. \square

Theorem 1.1 reduces the computation of secat_0 for fibrations with fibre X satisfying Halperin's Conjecture to the question of the existence of a section. Stanley expressed the obstruction to a section in cohomological terms when $X = S^{2n}$ [16, Th.3.3]. We follow his approach to obtain the following example.

Example 3.5. Write $\text{Fib}_{\mathbb{C}P^m}(\mathbb{C}P^n)$ for the set of fibrations $\mathbb{C}P^m \rightarrow E \xrightarrow{p} \mathbb{C}P^n$ modulo rational fibre-homotopy equivalence. We assume $m < n$. The identity $B\text{aut}_1(\mathbb{C}P^m) \simeq_{\mathbb{Q}} \prod_{k=1}^m K(\mathbb{Q}, 2k)$ follows from [15, Pro.2.6(iii)]. By universality,

$$\text{Fib}_{\mathbb{C}P^m}(\mathbb{C}P^n) \equiv [\mathbb{C}P^n, B\text{aut}_1(\mathbb{C}P^m)_{\mathbb{Q}}] \equiv \oplus_{i=1}^m H^{2k}(\mathbb{C}P^n; \mathbb{Q}) \equiv \mathbb{Q}^m.$$

We associate the rational fibre-homotopy equivalence class of p with an explicit m -tuple $(a_{m-1}, \dots, a_0) \in \mathbb{Q}^m$ defined as follows. The relative Sullivan model for p is an inclusion $\wedge(x_2, y_{2n+1}; d) \rightarrow \wedge(x_2, y_{2n+1}) \otimes \wedge(u_2, v_{2m+1}), D$ with $Dx = dx = 0, Dy = dy = x^{n+1}, Du = 0$. As for Dv , we have

$$Dv = u^{m+1} + a_m u^m x + a_{m-1} u^{m-1} x^2 + \dots + a_0 x^{m+1}$$

for some $a_i \in \mathbb{Q}$. The basis change $u \mapsto u - \frac{1}{m+1} a_m x$ will depress the polynomial and we may assume $a_m = 0$.

Given a section $S: \wedge(x, y) \otimes \wedge(u, v) \rightarrow \wedge(x, y)$ write $S(u) = qx$ for $q \in \mathbb{Q}$. Comparing coefficients of x^{m+1} in the equation $S(Dv) = DS(v) = 0$ we see that $z = q$ is a solution to $z^{m+1} + a_{m-1} z^{m-1} + \dots + a_1 z + a_0 = 0$. The converse follows similarly and we obtain:

$$\text{secat}_0(p) = \begin{cases} 0 & \text{if } z^{m+1} + a_{m-1} z^{m-1} + \dots + a_1 z + a_0 \text{ has a rational root} \\ 1 & \text{otherwise.} \end{cases}$$

We conclude with a question arising from our work. Meier [15] and others have given various equivalent versions of Halperin's Conjecture. It would be interesting to have an equivalent version of the conjecture phrased in terms of the sectional category of the universal fibration. We pose the following:

Question 3.6. Let X be an F_0 -space. Does $\text{secat}_0(u_X) = 1$ imply X satisfies Halperin's Conjecture?

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